A Note on Foundations of Fuzzy Sets

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1. INTRODUCTION

Fuzzy sets allow us to model the semantics of vague
notions. The structure of a fuzzy set itself contains, how-
ever, some problems which have not been definitively solved,
yet. In this paper some of these problems and their solution
are outlined.

2. THE NOTION OF A FUZZY SET

Every vague notion defines certain class of objects
whose boundaries we are not able to determine. Taking one
element of the universe by another and trying to recognize
it as an element of such a class, we soon meet with tremen-
dous problems. The elements of this class cannot be put down
into any list and, hence, do not form a set.

The basic idea of the notion of fuzzy set is as follows:
as we are not able to determine the exact boundaries of the
class defined by a vague notion, let us replace the decision
if an element belongs to it or not by a measure from any sca-
le. The smaller measure expresses that the element given is
somewhat closer to the edge of the class.

<u>Definition 1</u>: Let U be a universe and $\mathcal{L} = \langle L, \vee, \wedge, \underline{1}, \underline{0} \rangle$
a lattice. Then a fuzzy set A in the universe U, $A \subseteq U$, is
a function

$$A: U \rightarrow L \ .$$

The fuzzy set is identified with its membership function

in this definition. The formula $x \in_\alpha A$ means that x is an element of A in the grade $\alpha \in L$ or, equivalently, "the grade of membership of x in A is at least α", i.e. $Ax \geq \alpha$. From the point of fuzzy logic it is a predicate whose truth value is $\alpha \in L$.

The following presumption can be formulated on the basis of the results of fuzzy logic, psychological experiments, and intuition.

Presumption: The lattice $\mathcal{L}$ of the grades of membership is complete, infinitely distributive, and residuated lattice $\mathcal{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, \underline{1}, \underline{0} \rangle$.

For the properties of the residuated lattice see [3]. The lattice $I = \langle \langle 0, 1 \rangle, \vee, \wedge, \oplus, \rightarrow, 1, 0 \rangle$ is residuated lattice in which $\vee, \wedge$ are max and min operations, product is an operation

$$(1) \qquad \alpha \oplus \beta = 0 \vee (\alpha + \beta - 1)$$

and residuum is an operation

$$(2) \qquad \alpha \rightarrow \beta = 1 \wedge (1 - \alpha + \beta).$$

Fuzzy logic defined on this lattice is complete (see [3]). Since Boolean lattice $\langle \{true, false\}, \vee, \wedge, ', true, false \rangle$ is a special case of residuated lattice in which $\rightarrow$ is identified with classical implication and $\oplus$ with conjunction, the membership function of a fuzzy set can be understood to be a generalization of characteristic function of classical set.

3. THE STRUCTURE OF FUZZY SET

The improtant representation theorem of fuzzy sets can be proved:

Theorem 1: Every fuzzy set $A: U \rightarrow L$ can be identified

with a sequence of its α-cuts

$$\mathcal{A}_L = \left\{ A_\alpha;\ \alpha \in L \right\}$$

if we define

$$Ax = \bigvee \left\{ \alpha;\ x \in A_\alpha \right\}$$

for each $x \in U$.

By this theorem every fuzzy set can be viewed as a system of classical sets being a sequence in which each set is adjoined a subscript $\alpha \in L$. If $\mathcal{L}$ is a chain then this sequence is nonincreasing and every element is adjoined the degree of membership equal to the greatest subscript of the set to which it still belongs.

Let us consider some property φ (e.g. "to be red"). This property determines a class

(3) $$X = \left\{ x;\ \varphi(x) \right\}$$

which is not, however, a set since not every element can be uniquely decided if it has the property φ .On the other hand, there exists a set U for which

(4) $$X \subseteq U$$

holds (it can be the set of wave lengths ranging from 300 nm - 800 nm in our case). The relation (4) defines X to be the semiset, i.e. the proper class being a subclass of some set (see [5]). It is aparent that semiset has the same meaning as fuzzy set. It is thus an imaginary class having been inserted in U. Similarly, vague notion is imaginary, too. No exact function was used in its definition (apart from the definition of fuzzy set) and, therefore, semisets can be considered to be more realistic model of vagueness than fuzzy sets.

It was shown in [2] that fuzzy sets can be reasonable approximation of semisets, and so they can become technical means for work with them. There was also discussed the problem of determination of the kernel of fuzzy set, i.e. the set of prototypes (typical examples of elements having the property). We shall further suppose that the kernel $K \subseteq X \subseteq$ $\subseteq U$ is known.

Consider an element x having some property. Let y has a property which is similar to the property of the element x. Then we can express the similarity of both these properties by the formula

$$(5) \qquad x \equiv_\alpha y$$

stating that x is equal to y in some degree. The relation $\equiv_\alpha$ is called fuzzy identity and the degrees $\alpha, \beta \cdots$ are supposed to be elements of the residuated lattice $\mathcal{L}$. For the properties of fuzzy identity see [4]. The pair $\langle U, \equiv_\alpha \rangle$ where $\equiv_\alpha$ is a fuzzy identity on U is called F-space.

Definition 2: Let U be a universe and $\mathcal{L}$ the residuated lattice. Nebula is a three-tuple $\langle \langle U, \equiv_\alpha \rangle, K, R \rangle$ where $\langle U, \equiv_\alpha \rangle$ is F-space, K kernel and $R \subseteq U$ the range of nebula.

Nebula represents structured fuzzy set (see [4]). Let us suppose that there exists a set of set-formulas $sf_\varphi =$
$$= \{ \psi_\alpha \; ; \; \varphi \rightarrow \psi_\alpha \; \& \; \alpha = 1, 2, \cdots, p \} \, .$$

Definition 3: Two elements x, y are φ-similar, $x \sim y$, if there exists a set $\Delta \subseteq sf_\varphi$ such that for each $\psi \in \Delta$ $\psi(x) \, \& \, \psi(y)$ holds.

Intuitively, wo elements are similar if there are some properties common to both of them. Elements having the property φ surely have all the properties $\psi \in sf_\varphi$. If two ele-

ments are φ-similar then the number m_Δ (the number of elements of $\Delta \subseteq sf_\varphi$) is the degree of their similarity. It can be verified that the relation of φ-similarity is a fuzzy identity. The lattice of grades of membership in this case is the chain $\{0 < 1 < \ldots < p\}$. Then the three-tuple

$$\langle\langle U, \sim_n\rangle , K, A_1\rangle$$

where $\sim_n$ is the relation of φ-similarity and $A_1 = \{x \in U;$ $x \sim_1 y$ & $y \in K\}$ is nebula (hence, a fuzzy set).

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